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# Two-parameter ladder operators for spherical Bessel functions

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ABSTRACT

We construct ladder operators for spherical Bessel functions of arbitrary order. Our ladder operators act independently on two parameters, one of which is the order of the spherical Bessel function, while the other parameter is a multiplicative factor in the spherical Bessel function's argument.



## 1. Introduction

Ladder operators are usually defined on function spaces, the elements of which are associated with a numerical parameter. Application of the ladder operator to a function then maintains the function's form, while raising or lowering the value of its parameter. Famous examples of ladder operators are found in Quantum Mechanics, where they raise or lower the principal quantum number of functions within a discrete set of Schrödinger eigenstates. This way all eigenstates can be obtained from a single eigenstate, just by repetitive application of a ladder operator [1]. There is a vast amount of literature on ladder operators and its applications, e.g. to the Schrödinger eigenstates of Kratzer and Morse potentials [2], to angular momenta [3,4], to trigonometric functions [5], to shape-invariant potentials [6], to Szego polynomials, [7], to the Dirac equation with Coulomb potential [8] and many more. The purpose of the present paper is to construct ladder operators for spherical Bessel functions, which arise in many fields of physics and engineering. Besides being solutions of the Schrödinger equation for the infinite, radially symmetric potential well [1], applications of spherical Bessel functions include inverse scattering theory [9], antenna pattern synthesis [10], acoustics [11] and more [12]. Moreover, a special case of spherical Bessel functions is given by the sinc function, which has a large amount of applications in signal processing, for an introduction and basic properties the reader may refer to [13]. Spherical Bessel functions are characterized by their order [14], a nonnegative integer, which we will take as a parameter for the ladder operators to be constructed in this paper. Furthermore, we will introduce a real multiplicative constant in the argument of the spherical Bessel function and take this constant as a second parameter, such that the ladder operator acts independently on each of the two parameters. In the remainder of this paper, Section 2 is devoted to the construction of our ladder operators, an application of which will be presented in Section 3.

## 2. Ladder operators for spherical Bessel functions

Let $m \in \mathbb{R}^+$ and let $V_m$ be the vector space $V_m := \operatorname{span}(\sin(mx), \cos(mx))$. For $\mu, \nu \in \mathbb{R}^+$ with $\mu > \nu$ the ladder operator $A_{\mu,\nu}$ for trigonometric functions is then defined as the following map [5]:

$$A_{\mu,\nu} : V_\mu \to V_{\mu+\nu}, \quad A_{\mu,\nu}(v) := \left[\frac{\sin(\nu x)}{\mu}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(\nu x)\right](v). \tag{1}$$

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Thus, the ladder operator moves vectors between the different spaces of type $V_m$. In particular we have

$$A_{\mu,\nu}[\sin(\mu x)] = \sin[(\mu+\nu)x], \qquad A_{\mu,\nu}[\cos(\mu x)] = \cos[(\mu+\nu)x].$$

In order to extend the concept of ladder operators, we note the following connection between trigonometric functions and spherical Bessel functions [14]

$$\sin(mx) = mxj_0(mx) \qquad \cos(mx) = -mxy_0(mx), \tag{2}$$

where $j_0, y_0$ denote the zero-order spherical Bessel functions of first and second kind, respectively. Thus, we can also apply the ladder operator (1) to the spherical Bessel functions of order zero:

$$A_{\mu,\nu}[xj_0(\mu x)] = \frac{1}{\mu}A_{\mu,\nu}[\sin(\mu x)] = \frac{1}{\mu}\sin[(\mu+\nu)x] = \frac{\mu+\nu}{\mu}xj_0[(\mu+\nu)x].$$

In the same way we obtain

$$A_{\mu,\nu}[xy_0(\mu x)] = \frac{\mu+\nu}{\mu}xy_0[(\mu+\nu)x].$$

Note that the minus sign on the cosine term in (2) does not matter, as it cancels out by twofold application of (2). As is well known [12], both of these spherical Bessel functions of order $n \in \mathbb{N}\cup\{0\}$ satisfy the same recurrence relations, namely, for $k_n \in \{j_n, y_n\}$ we have

$$xk_n(mx) = \frac{1}{m}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n}{x}\right)xk_{n-1}(mx) \tag{3}$$

$$xk_{n-1}(mx) = \frac{1}{m}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n}{x}\right)xk_n(mx). \tag{4}$$

Let us now construct ladder operators for the spherical Bessel functions. To this end, we combine (1), (3) and (4), considering the case $n = 1$:

$$\begin{aligned} xk_1[(\mu+\nu)x] &= \frac{1}{\mu+\nu}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right)xk_0[(\mu+\nu)x] \\ &= \frac{\mu}{(\mu+\nu)^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right)A_{\mu,\nu}[xk_0(\mu x)] \\ &= \frac{\mu}{(\mu+\nu)^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right)A_{\mu,\nu}\left[\frac{1}{\mu}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)xk_1(\mu x)\right] \\ &= \frac{1}{(\mu+\nu)^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right)A_{\mu,\nu}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{x}\right)xk_1(\mu x)\right]. \end{aligned}$$

This process can be iterated to deliver the general case:

$$\begin{aligned} xk_n((\mu+\nu)x) &= \frac{1}{(\mu+\nu)^n}\prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)xk_0[(\mu+\nu)x] \\ &= \frac{\mu}{(\mu+\nu)^{n+1}}\prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)A_{\mu,\nu}[xk_0(\mu x)] \\ &= \frac{\mu^{n-1}}{(\mu+\nu)^{n+1}}\prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)A_{\mu,\nu}\left[\prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{j}{x}\right)xk_n(\mu x)\right]. \end{aligned} \tag{5}$$

On employing the explicit form of $A_{\mu,\nu}$ as given in (1), we summarize our above findings as follows: for $m \in \mathbb{R}^+$ and $n \in \mathbb{N}$ let $V_m^n$ be the vector space $V_m^n := \operatorname{span}(xj_n(mx), xy_n(mx))$. For $\mu, \nu \in \mathbb{R}^+$ with $\mu > \nu$ and $k \in \mathbb{N}$ the ladder operator $\mathcal{A}_{\mu,\nu}$ for spherical Bessel functions is given as follows:

$$\mathcal{A}_{\mu,\nu}\colon V_\mu^k \to V_{\mu+\nu}^k,$$

$$\mathcal{A}_{\mu,\nu}(v) := \frac{\mu^{k-1}}{(\mu+\nu)^{k+1}}\left[\prod_{j=1}^{k}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)\right]\left[\frac{\sin(\nu x)}{\mu}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(\nu x)\right]\left[\prod_{j=1}^{k}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{j}{x}\right)\right](v). \tag{6}$$

Hence, this differential operator of order $2k+1$ maps functions of spherical Bessel type multiplied by the independent variable onto themselves, retaining their order while changing their argument's factor. From the above considerations we can also define a ladder operator $\mathcal{A}_{\mu,\nu}^k$ that connects the zero-order spherical Bessel functions with their $n$th order counterparts: using the same parameter settings as for $\mathcal{A}_{\mu,\nu}$ and the explicit form of $A_{\mu,\nu}$ as given in (1), we define by means of (5)

$$\mathcal{A}^k_{\mu,\nu}: V_\mu \to V^k_{\mu+\nu},$$

$$\mathcal{A}^k_{\mu,\nu}(v) := \frac{\mu^{k-1}}{(\mu+\nu)^{k+1}} \left[\prod_{j=1}^{k} \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)\right] \left[\frac{\sin(\nu x)}{\mu}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(\nu x)\right](v).$$

This differential operator of order $k+1$ is defined on $V_\mu$, that is, zero-order spherical Bessel functions. It raises their order to $k$ and changes their argument's factor.

## 3. Application

Consider the stationary Schrödinger equation in atomic units

$$\psi''(x) + \left[E - \frac{k(k+1)}{x^2}\right]\psi(x) = 0, \tag{7}$$

where $E \in \mathbb{R}, E \neq 0$ is the stationary energy and $k \in \mathbb{N} \cup \{0\}$ is arbitrary. We impose the following Dirichlet boundary conditions:

$$\psi(0) = \psi(1) = 0. \tag{8}$$

In the simplest case $k = 0$, Eq. (7) has the general solution

$$\psi(x) = a\sin\left(\sqrt{E}x\right) + b\cos\left(\sqrt{E}x\right),$$

where $a, b \in \mathbb{R}$. This means that for fixed $E$ we have $\psi \in V_{\sqrt{E}}$. Now, taking into account the boundary conditions (8), we obtain the discrete energy spectrum $(E_\mu)$, the explicit form of which reads

$$E_\mu = \pi^2\mu^2,$$

where $\mu \in \mathbb{N}$. The corresponding solution set $(\psi_\mu) \subset \bigcup_{\mu\in\mathbb{N}} V_{\pi\mu}$ is given by

$$\psi_\mu(x) = \sin(\pi\mu x). \tag{9}$$

Now we proceed to the case $k = 1$. To this end, note that according to (2), the solution (9) can also be written in the form

$$\psi_\mu(x) = xj_0(\pi\mu x), \tag{10}$$

where we omitted an irrelevant constant. In order to obtain the solution of our problem (7)–(8), we now make use of our ladder operator (6) by applying it to the solution (10). To this end, let $z \neq 0$ be the smallest positive zero of the spherical Bessel function $j_1$ and let $\nu = z - \pi\mu$. We obtain

$$\begin{aligned} A^1_{\mu,\nu}(\psi_\mu) &= \frac{\mu}{(\mu+\nu)^2}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)A_{\mu,\nu}[xj_0(\pi\mu x)] \\ &= xj_1[(\pi\mu+\nu)x]. \end{aligned} \tag{11}$$

Let us abbreviate $\psi^1_\mu(x) = xj_1[(\pi\mu+\nu)x]$, then clearly $\psi^1_\mu \in V^1_{\pi\mu+\nu}$. This function is a solution of Eq. (7) for $k = 1$. Furthermore, Taylor expansion around zero gives

$$xj_1(x) = \frac{1}{3}x^2 + o\left(x^3\right),$$

which implies that $\psi^1_\mu$ vanishes at zero and thus satisfies the first boundary condition in (8). Furthermore, since $\nu = z - \pi\mu$, we have $\psi^1_\mu(1) = j_1(z) = 0$, which means that the second boundary condition is also fulfilled. Now let us generalize to the case of arbitrary $k$. To this end, we apply our ladder operator (6) to $\psi_\mu$, as given in (10). Let $z \neq 0$ be the smallest positive zero of the spherical Bessel function $j_k$ and let $\nu = z - \pi\mu$. We obtain

$$\begin{aligned} A^k_{\mu,\nu}(\psi_\mu) &= \left[\frac{\mu}{(\mu+\nu)^{k+1}}\prod_{j=1}^{k}\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{j}{x}\right)A_{\mu,\nu}\right][xj_0(\pi\mu x)] \\ &= xj_k[(\pi\mu+\nu)x]. \end{aligned} \tag{12}$$

We abbreviate $\psi^k_\mu(x) = xj_k[(\pi\mu+\nu)x]$. The function $\psi^k_\mu \in V^k_{\pi\mu+\nu}$ is a solution of Eq. (7) for arbitrary $k$. Furthermore, Taylor expansion around zero gives

$$xj_k(x) = \frac{\sqrt{\pi}}{2^{k+1}\Gamma\left(k+\frac{3}{2}\right)}x^{k+1} + o\left(x^{k+3}\right), \tag{13}$$

where $\Gamma$ stands for the Gamma function. Now, (13) implies that $\psi^k_\mu$ vanishes at zero and thus satisfies the first boundary condition in (8). Furthermore, since $\nu = z - \pi\mu$, we have $\psi^k_\mu(1) = j_k(z) = 0$, which means that the second boundary condition is also fulfilled.

## 4. Conclusions

We have constructed two-parameter ladder operators for spherical Bessel functions, acting on their order and on a multiplicative factor in their argument. Thus, any spherical Bessel function with arbitrary multiplicative factor in its argument can be generated by iterative application of our ladder operators to spherical Bessel functions of order zero.

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